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Box Spline Prewavelets of Small Support

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The purpose of this paper is the construction of bi- and trivariate prewavelets from box-spline spaces, i.e., piecewise polynomials of fixed degree on a uniform mesh. They have especially small support and form Riesz bases of the wavelet spaces, so they are stable. In particular, the supports achieved are smaller than those of the prewavelets due to Riemenschneider and Shen in a recent, similar construction.

1. INTRODUCTION

There are many useful ways to decompose uni- and multivariate functions for the purpose of analysing, classifying, transmitting, or filtering the signals represented by them. One such method which currently attracts a lot of attention in applications and in the approximation theory community is the (pre-)wavelets decomposition, which we describe below.

It is quite well understood how wavelets and prewavelets are generated in one dimension, especially if we think of spline (pre-)wavelets [4] or the so-called Daubechies wavelets [7]. However, there is still much work to do in more than one dimension. Of course, a tensor product approach can always be used, in particular in connection with spline wavelets, but as it is with tensor product B-spline bases versus the far superior box-splines, their supports are usually too large. This is highly relevant, for instance, if we use such bases as finite elements for Galerkin-type methods in order to

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solve partial differential equations with numerical methods, because they make it expensive to compute the entries of stiffness matrices.

Therefore we address the construction of bi- and trivariate box-spline prewavelets of small support in this note. To this end, we step back now and recall the definition of prewavelets and of the so-called multiresolution analysis (MRA) which is fundamental to the construction of all types of wavelets. The goal always is to decompose any $f \in L^2(\mathbb{R}^d)$, i.e., the square-integrable real (or complex) valued functions in d dimensions into orthogonal series of basis functions. For this we require to start with a multiresolution analysis, i.e., a nested sequence of closed subspaces $V_j \subset L^2(\mathbb{R}^d)$

$$\cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R}^d)$$

that satisfy the following three fundamental properties:

(i) $f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$ for all integers j,

(ii) suppose

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\}, \qquad \overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R}^d)$$

(but see [1] for conditions under which (ii) is redundant),

(iii) there is a Riesz basis $\{B(\cdot - k) \mid k \in \mathbb{Z}^d\}$ of V_0, i.e.,

$$V_0 = \operatorname{span}_{\ell^2(\mathbb{Z}^d)}\{B(\cdot - k) \mid k \in \mathbb{Z}^d\},$$

where the coefficients of the spanning functions are always square-summable as indicated by the subscript, and there exist positive and finite constants λ and Λ such that for all $c \in \ell^2(\mathbb{Z}^d)$

$$\lambda \, \|c\|_2 \leqslant \left\| \sum_{k \in \mathbb{Z}^d} c_k B(\cdot - k) \right\|_2 \leqslant \Lambda \, \|c\|_2.$$

Here we use the notation $c = (c_k)_{k \in \mathbb{Z}^d}$ and the 2-norms denote the Euclidean norm on $\ell^2(\mathbb{Z}^d)$ or on $L^2(\mathbb{R}^d)$ as is appropriate from the context.

The properties (i)–(iii) have many fundamental consequences. One of them is that if we can find a collection of square-integrable functions named *prewavelets* $\psi \in V_1 \setminus \{0\}$, $\psi \perp V_0$, call the set of prewavelets $\Psi \ni \psi$, such that the direct sum W of all

$$W_\psi := \operatorname{span}_{\ell^2(\mathbb{Z})}\{\psi(\cdot - k) \mid k \in \mathbb{Z}^d\}$$

as ψ varies over Ψ, forms the orthogonal complement of V_0 within V_1, then

$$L^2(\mathbb{R}^d) = \bigoplus_{j=-\infty}^{\infty} W_j,$$

where W_j denotes W with the functions scaled by 2^j. In other words,

$$W = \bigoplus_{\psi \in \Psi} W_\psi, \qquad W_j = \{g(2^j \cdot) \mid g \in W\}.$$

Therefore we have the desired decomposition of the whole of $L^2(\mathbb{R}^d)$, because the W_j are mutually orthogonal which follows from a standard argument using the fact that the prewavelets ψ are orthogonal to V_0 and from (i). We remark that it is well known that in this setting Ψ contains $2^d - 1$ elements.

Clearly, decompositions of this kind are efficient if the prewavelets are compactly supported, and the smaller support, the better localisation and the simpler their computation and application in filtering tools, etc. Therefore there have been several approaches to construct prewavelets of small support, most notably that of Riemenschneider and Shen [10], see also Chui *et al.* [5], which uses the box-splines that are familiar from the book [3] for instance. We will provide their formal definition as the inverse Fourier transforms of certain simple entire functions in the next section but just point out at this point that they are a multivariate generalisation of the famous univariate B-splines, i.e., piecewise polynomials of compact support that span multivariate spline spaces. The prewavelets of Riemenschneider and Shen work only in less than four dimensions and so do ours which are a development from their construction. Ours have smaller support however, especially if smoothness is required. Our work was partly motivated by the construction by Kotyczka and Oswald [8] of continuous piecewise linear prewavelets with small support in two dimensions. However, their construction is rather ad hoc and appears to have no generalisation to higher smoothness.

The notation and a very short introduction to box-splines are given at the beginning of the next section where the prewavelets are constructed as well.

2. A CONSTRUCTION OF BOX SPLINE PREWAVELETS IN $\mathbb{R}^d$, $d = 1, 2, 3$

Let $v_1, \ldots, v_\ell$ be different vectors in $\{-1, 0, 1\}^d = \mathbb{Z}^d \cap [-1, 1]^d$ which span $\mathbb{R}^d$ (linear independence is not required). The *box-spline* B associated with these vectors called directions with multiplicities $n_1, \ldots, n_\ell \geqslant 1$, respectively, may be defined by its Fourier transform

$$\hat{B}(u) := \left(\frac{1-z^{v_1}}{iv_1 u}\right)^{n_1} \cdots \left(\frac{1-z^{v_\ell}}{iv_\ell u}\right)^{n_\ell}, \qquad u=(u_1, \ldots, u_d) \in \mathbb{R}^d,$$

where $z=(z_1, \ldots, z_d)=(e^{-iu_1}, \ldots, e^{-iu_d})$, $z^{v_k} := z_1^{v_{1,k}} \cdots z_d^{v_{d,k}}$, and $v_k u$ denotes the scalar product of the two vectors, $v_{1,k}u_1 + \cdots + v_{d,k}u_d$. This Fourier transform is an entire function of exponential type and its inverse Fourier transform is of compact support. The $v_{j,k}$ are the components of the d-dimensional vectors v_k.

We have further the so-called refinement equation in Fourier transform form

$$\hat{B}(2u) = 2^{-n} H(z)\, \hat{B}(u),$$

where $n := \sum_{k=1}^{\ell} n_k$, and

$$H(z) := (1+z^{v_1})^{n_1} \cdots (1+z^{v_\ell})^{n_\ell}. \tag{2.1}$$

The refinement equation in this form is simple to derive by using the definition of $\hat{B}$. If we take inverse Fourier transforms on both sides in the penultimate display, we get that the spaces spanned by scales of the translates of the functions B—as in Section 1 the spaces V_j—satisfy condition (i) of the requirements on a multiresolution analysis. More concretely, we get an expression for B in terms of translates of its scaled version $B(2\cdot)$. It is well known that the MRA's other requirements hold too for box-splines, where for (iii) there is an additional condition on the directions required which we explain now and which has the required Riesz conditions as a consequence. To this end, let

$$P(z) := \sum_{j \in \mathbb{Z}^d} (B * B(-\cdot))(j)\, z^j = \sum_{j \in \mathbb{Z}^d} |\hat{B}(u+2\pi j)|^2,$$

where u is as above. If the matrix

$$[v_1, \ldots, v_\ell] = \begin{bmatrix} v_{1,1} & \cdots & v_{1,\ell} \\ \vdots & \ddots & \vdots \\ v_{d,1} & \cdots & v_{d,\ell} \end{bmatrix}$$

is *unimodular*, i.e., $|\det X| = 1$ for any non-singular $d \times d$ submatrix X, then

$$P(z) > 0 \qquad \text{for all} \quad u \in [-\pi, \pi]^d \tag{2.2}$$

(Dahmen and Micchelli [6]). This implies (iii), where in fact λ and Λ are the minimum and the maximum of the periodic function $\sqrt{P(z)}$, respectively. Let $\mathscr{V}$ denote the set of vertices of the d-dimensional unit cube $[0, 1]^d$. Define ψ_j, $j \in \mathscr{V} \setminus \{0\}$, by its Fourier transform

$$\hat{\psi}_j(2u) = G_j(z)\, \hat{B}(u),$$

where the function G_j is to be specified later. Then for $j \in \mathscr{V} \setminus \{0\}$, ψ_j lies in the prewavelet space, i.e., $\psi_j \in V_1$ and is orthogonal to V_0 with respect to the standard Euclidean inner product of square-integrable functions in the sense of the introduction, if

$$P(z)\, H(z)\, \overline{G_j(z)} + P((-1)^j z)\, H((-1)^j z)\, \overline{G_j((-1)^j z)} \equiv 0. \tag{2.3}$$

Here we use the notation $(-1)^j z := ((-1)^{j_1} z_1, \ldots, (-1)^{j_d} z_d)$ with $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$. Moreover, the multiinteger translates $\psi_j(\cdot - k)$, $j \in \mathscr{V} \setminus \{0\}$, $k \in \mathbb{Z}^d$, form a Riesz basis of the space $W \subset V_1$ which is the orthogonal complement of V_0 if and only if the $(2^d - 1) \times 2^d$ matrix

$$N := [G_j((-1)^k z)]_{j \in \mathscr{V} \setminus \{0\},\, k \in \mathscr{V}}$$

has full rank for $|z_1| = \cdots = |z_d| = 1$. We wish to construct functions G_j such that this is the case. Indeed, Riemenschneider and Shen [10] prove in Proposition 3.6 and Corollary 3.7 that under this rank condition, the aforementioned multiinteger translates of ψ_j, $j \in \mathscr{V} \setminus \{0\}$, *together* with $B(\cdot - k)$, $k \in \mathbb{Z}^d$, form a Riesz basis of the whole space V_1 that we wish to decompose as $V_1 = V_0 + W$, see also Ron and Shen [2, 11, 12]. Thus it follows from the orthogonality of the ψ_j to V_0 that the prewavelets and their translates alone form a Riesz basis of W. We now proceed to constructing G_j so that both (2.3) and the above rank condition hold.

To begin with, let η: $\mathscr{V} \to \mathscr{V}$ satisfy $\eta(0) = 0$ and

$$(\eta(\mu) + \eta(\nu))(\mu + \nu) \qquad \text{is odd for} \quad \mu \neq \nu. \tag{2.4}$$

Examples of such an η for dimensions $d = 1, 2, 3$, as well as a remark that no mapping with this property exists for $d > 3$, can be found in [9, 10]. The existence of this function is decisive for the whole construction.

Now define for $j \in \mathscr{V} \setminus \{0\}$, recalling the definition of P from the above,

$$\overline{G_j(z)} = z^{\eta(j)} P((-1)^j z) \prod_{\substack{k=1 \\ v_k j \text{ odd}}}^{\ell} (1 - z^{v_k})^{n_k} \prod_{\substack{k=1 \\ v_k j \text{ even}}}^{\ell} S_k(z^{v_k}), \tag{2.5}$$

where the univariate Laurent polynomials S_k, $k = 1, \ldots, \ell$, are yet to be chosen.

Noting that $((-1)^j z)^{v_k} = (-1)^{v_k j} z^{v_k}$ and $((-1)^j z)^{\eta(j)} = (-1)^{j\eta(j)} z^{\eta(j)} = -z^{\eta(j)}$, we see that the G_j defined by (2.5) satisfy condition (2.3) regardless of any particular choice of S_k, $k = 1, \ldots, \ell$. Our special choice of the latter will only be needed in order to fulfill the aforementioned rank condition.

Defining $G_0(z) := P(z)\, H(z)$, we introduce the matrix

$$\hat{N} := [\overline{G_j((-1)^k z)}]_{j,k \in \mathscr{V}}.$$

We shall choose $S_1, \ldots, S_\ell$ so that $\hat{N}$ is non-singular and hence N has full rank everywhere on the unit sphere, which is the required rank property. In order to evaluate the determinant more easily, we decompose the matrix as follows. We let

$$M := [G_0((-1)^j z)\, \delta_{j,k}]_{j,k \in \mathscr{V}}, \qquad \tilde{M} := \left[z^{\eta(j)} \prod_{\substack{i=1 \\ v_i j \text{ odd}}}^{\ell} (1 - z^{2v_i})^{n_i}\, \delta_{j,k} \right]_{j,k \in \mathscr{V}}.$$

Then we have the matrix decomposition

$$\hat{N} M = \tilde{M} A,$$

where we use the $2^d \times 2^d$-matrix $A = [A_{j,k}]_{j,k \in \mathscr{V}}$, with

$$A_{0,k} = |G_0((-1)^k z)|^2, \qquad k \in \mathscr{V},$$

and for $j \in \mathscr{V} \setminus \{0\}$, $k \in \mathscr{V}$, with the matrix entries

$$\begin{aligned} A_{j,k} = {} & (-1)^{k\eta(j)}\, P((-1)^{j+k} z)\, P((-1)^k z) \\ & \times \prod_{\substack{i=1 \\ v_i j \text{ even}}}^{\ell} (1 + (-1)^{v_i k} z^{v_i})^{n_i}\, S_i((-1)^{v_i k} z^{v_i}). \end{aligned}$$

Now the determinant of the diagonal matrix M is easy to evaluate as the product

$$\det M = \prod_{j \in \mathscr{V}} P((-1)^j z)\, H((-1)^j z),$$

and, moreover, the determinant of $\tilde{M}$ is the product

$$\begin{aligned} \det \tilde{M} &= z^\zeta \prod_{i=1}^{\ell} \prod_{\substack{j \in \mathscr{V} \\ v_i j \text{ odd}}} (1 - z^{2v_i})^{n_i} = z^\zeta \prod_{i=1}^{\ell} (1 - z^{2v_i})^{2^{d-1} n_i} \\ &= z^\zeta \prod_{j \in \mathscr{V}} H((-1)^j z), \end{aligned}$$

where $\zeta := \sum_{j \in \mathscr{V}} \eta(j)$. Therefore, using the decomposition above,

$$z^{\zeta} \det A = \prod_{j \in \mathscr{V}} P((-1)^j z) \det \hat{N}.$$

Therefore it suffices to show that A is non-singular so that the desired result of full rank follows.

Now we observe that for each $j \in \mathscr{V} \setminus \{0\}$, $\sum_{k \in \mathscr{V}} A_{j,k} = 0$. Thus the $(2^d - 1) \times (2^d - 1)$ signed minors of A taken from these rows are equal and we can evaluate the determinant of A as

$$\det A = \left(\sum_{k \in \mathscr{V}} A_{0,k} \right) \det[A_{j,k}]_{j,k \in \mathscr{V} \setminus \{0\}}.$$

We have, moreover, by (2.2)

$$\sum_{k \in \mathscr{V}} A_{0,k} = \sum_{k \in \mathscr{V}} |G_0((-1)^k z)|^2 > 0$$

and

$$\det[A_{j,k}]_{j,k \in \mathscr{V} \setminus \{0\}} = \prod_{k \in \mathscr{V} \setminus \{0\}} P((-1)^k z) \det \tilde{A},$$

where $\tilde{A}$ is the $(2^d - 1) \times (2^d - 1)$ matrix such that for $j, k \in \mathscr{V} \setminus \{0\}$,

$$\tilde{A}_{j,k} = (-1)^{k\eta(j)} P((-1)^{j+k} z) \prod_{\substack{i=1 \\ v_i j \text{ even}}}^{\ell} (1 + (-1)^{v_i k} z^{v_i})^{n_i} S_i((-1)^{v_i k} z^{v_i}).$$

It remains to choose $S_1, \ldots, S_\ell$ so that $\tilde{A}$ is non-singular to complete our construction such that the Riesz basis property is guaranteed. To this end we require two additional auxiliary results.

LEMMA 2.1. *For any function $Q(z)$, let C be the $(2^d - 1) \times (2^d - 1)$ matrix given by*

$$C_{j,k} = (-1)^{k\eta(j)} Q((-1)^{j+k} z), \qquad j, k \in \mathscr{V} \setminus \{0\}.$$

Then its determinant is the product

$$\det C = -Q(z) \left\{ \sum_{j \in \mathscr{V}} Q((-1)^j z)^2 \right\}^{2^{d-1} - 1}.$$

Proof. Let D be the $2^d \times 2^d$ matrix given by

$$D_{j,k} = (-1)^{k\eta(j)} Q((-1)^{j+k} z), \qquad j, k \in \mathscr{V}.$$

Since the rows of D are orthogonal we have

$$DD^T = \left\{ \sum_{j \in \mathscr{V}} Q((-1)^j z)^2 \right\} I_{2^d},$$

where I_{2^d} denotes the $2^d \times 2^d$ identity matrix. Noting that $D_{0,0} = Q(z)$ and

$$D_{j,j} = -Q(z), \qquad j \in \mathscr{V} \setminus \{0\},$$

we have the determinant

$$\det D = -\left\{ \sum_{j \in \mathscr{V}} Q((-1)^j z)^2 \right\}^{2^{d-1}}.$$

Now let E be the product of D times a diagonal matrix

$$E := D[Q((-1)^j z)\, \delta_{j,k}]_{j,k \in \mathscr{V}}.$$

Then all the rows of E except for $j = 0$ sum to zero, and we have, as for the matrix A above,

$$\det E = \left\{ \sum_{j \in \mathscr{V}} Q((-1)^j z)^2 \right\} \prod_{j \in \mathscr{V} \setminus \{0\}} Q((-1)^j z) \det C.$$

On the other hand, the determinant is

$$\begin{aligned} \det E &= \prod_{j \in \mathscr{V}} Q((-1)^j z) \det D \\ &= -\prod_{j \in \mathscr{V}} Q((-1)^j z) \left\{ \sum_{j \in \mathscr{V}} Q((-1)^j z)^2 \right\}^{2^{d-1}}, \end{aligned}$$

and the result follows. ∎

LEMMA 2.2. *For any continuous functions $Q(z)$, $f_1(z)$, ..., $f_\ell(z)$, let C be the $(2^d - 1) \times (2^d - 1)$ matrix given by*

$$C_{j,k} = (-1)^{k\eta(j)}\, Q((-1)^{j+k} z) \prod_{\substack{i=1 \\ v_i j \text{ even}}}^{\ell} f_i((-1)^{v_i k} z^{v_i}), \qquad j, k \in \mathscr{V} \setminus \{0\}.$$

Then its determinant is

$$\det C = -Q(z) \left\{ \sum_{j \in \mathscr{V}} (Q((-1)^j z))^2 \prod_{i=1}^{\ell} f_i((-1)^{v_i j} z^{v_i}) \right\}^{2^{d-1}-1}.$$

Proof. We may, without loss of generality, assume that $f_1, ..., f_\ell > 0$. Let X, Y be diagonal $(2^d-1)\times(2^d-1)$ matrices defined by

$$X_{j,k} = \delta_{j,k} \prod_{\substack{i=1 \\ v_i j \text{ odd}}}^{\ell} f_i(z^{v_i})^{1/2} f_i(-z^{v_i})^{1/2}, \qquad j, k \in \mathscr{V} \setminus \{0\},$$

$$Y_{j,k} = \delta_{j,k} \prod_{i=1}^{\ell} f_i((-1)^{v_i j+1} z^{v_i})^{1/2}, \qquad j, k \in \mathscr{V} \setminus \{0\}.$$

Then for $j, k \in \mathscr{V} \setminus \{0\}$, we have the product for each entry of the matrix XCY

$$(XCY)_{j,k} = (-1)^{k\eta(j)} Q((-1)^{j+k} z) \times \prod_{i=1}^{\ell} f_i((-1)^{v_i(j+k)} z^{v_i}) f_i((-1)^{v_i(j+k)+1} z^{v_i})^{1/2}.$$

Applying Lemma 2.1 with $Q(z)$ replaced by $Q(z) \prod_{i=1}^{\ell} f_i(z^{v_i}) f_i(-z^{v_i})^{1/2}$ gives the product determinant

$$\begin{aligned} \det(XCY) &= -Q(z) \prod_{i=1}^{\ell} f_i(z^{v_i}) f_i(-z^{v_i})^{1/2} \\ &\quad \times \left\{ \sum_{j\in\mathscr{V}} (Q((-1)^j z))^2 \prod_{i=1}^{\ell} f_i((-1)^{v_i j} z^{v_i})^2 f_i((-1)^{v_i j+1} z^{v_i}) \right\}^{2^{d-1}-1} \\ &= -Q(z) \prod_{i=1}^{\ell} f_i(z^{v_i})^{2^{d-1}} f_i(-z^{v_i})^{2^{d-1}-1/2} \\ &\quad \times \left\{ \sum_{j\in\mathscr{V}} (Q((-1)^j z))^2 \prod_{i=1}^{\ell} f_i((-1)^{v_i j} z^{v_i}) \right\}^{2^{d-1}-1} \end{aligned}$$

Since

$$\det X = \prod_{i=1}^{\ell} f_i(z^{v_i})^{2^{d-2}} f_i(-z^{v_i})^{2^{d-2}},$$

$$\det Y = \prod_{i=1}^{\ell} f_i(z^{v_i})^{2^{d-2}} f_i(-z^{v_i})^{2^{d-2}-1/2},$$

the result follows. ∎

So from Lemma 2.2 we have the determinant of $\tilde{A}$

$$\det \tilde{A} = -P(z) \left\{ \sum_{j\in\mathscr{V}} P((-1)^j z)^2 \prod_{i=1}^{\ell} (1+(-1)^{v_i j} z^{v_i})^{n_i} S_i((-1)^{v_i j} z^{v_i}) \right\}^{2^{d-1}-1}.$$

If $n_i = 2s$, we choose $S_i(z^{v_i}) = (z^{v_i})^{-s}$. Since for any $z \in \mathbb{C}$ with $|z| = 1$, we have the important identity

$$(1+z)^{2s} = |1+z|^{2s}\, z^s,$$

we have

$$(1+(-1)^{v_i j}\, z^{v_i})^{n_i}\, S_i((-1)^{v_i j}\, z^{v_i}) = |1+(-1)^{v_i j}\, z^{v_i}|^{n_i}.$$

If $n_i = 2s-1$, we choose $S_i(z^{v_i}) = (z^{v_i})^{-s}\,(1+z^{v_i})$. Then

$$(1+(-1)^{v_i j}\, z^{v_i})^{n_i}\, S_i((-1)^{v_i j}\, z^{v_i}) = |1+(-1)^{v_i j}\, z^{v_i}|^{n_i+1}.$$

Thus, for $i = 1, ..., \ell$,

$$S_i(z^{v_i}) := \begin{cases} (z^{v_i})^{-n_i/2}, & \text{if } n_i \text{ is even,} \\ (z^{v_i})^{-\lceil n_i/2 \rceil}\,(1+z^{v_i}), & \text{if } n_i \text{ is odd,} \end{cases} \tag{2.6}$$

and

$$\det \tilde{A} = -P(z) \left\{ \sum_{j \in \mathscr{V}} P((-1)^j\, z)^2 \prod_{i=1}^{\ell} |1+(-1)^{v_i j}\, z^{v_i}|^{m_i} \right\}^{2^{d-1}-1} < 0,$$

where $m_i = n_i$ or $n_i + 1$. Therefore we have the required property, namely that the prewavelets and their translates form a Riesz basis of W.

3. COMPARISON TO RIEMENSCHNEIDER–SHEN PREWAVELETS

The Riemenschneider–Shen prewavelets ψ_j, $j \in \mathscr{V} \setminus \{0\}$, are defined by their Fourier transforms

$$\hat{\psi}_j(2u) = H_j(z)\, \hat{B}(u),$$

with

$$H_j(z) = z^{\eta(j)} P((-1)^j\, z)\, H((-1)^j\, z), \tag{3.1}$$

where H is given by (2.1) and has to be replaced by $\bar{H}$ in the above display under certain parity conditions. Comparing $H_j(z)$ with $G_j(z)$ defined by (2.5) and (2.6) we note that significantly less of the translates of the box-spline are needed in our construction of prewavelets as in the Riemenschneider–Shen construction and therefore their support is smaller. This is because the multiplication by the various factors included in the S_i mainly introduces a shift of the whole function, while there are several factors of

H omitted in the definition of the G_js as compared to the H_j above. The gain becomes larger when there are larger multiplicities in the directions of the defining box-spline. In the case $d=1$, however, $H_j(z)$ and $G_j(z)$ are essentially the same and both lead to the univariate compactly supported pre-wavelets by Chui and Wang [4].

For example, let us consider in detail the case of the box-spline in two variables associated with the directions

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with multiplicities $n_1 = n_2 = n_3 = 2$. We have

$$H(z) = (1+z_1)^2\,(1+z_2)^2\,(1+z_1 z_2)^2.$$

Choosing η as in [10], we have by (3.1),

$$H_j(z) = P((-1)^j z) \begin{cases} z_1 z_2 (1-z_1)^2\,(1+z_2)^2\,(1-z_1 z_2)^2 & \text{if } \; j=(1,0), \\ z_2 (1-z_1)^2\,(1-z_2)^2\,(1-z_1 z_2), & \text{if } \; j=(0,1), \\ z_1 (1-z_1)^2\,(1-z_2)^2\,(1+z_1 z_2)^2, & \text{if } \; j=(1,1). \end{cases}$$

In our construction, by (2.5) and (2.6),

$$\overline{G_j(z)} = P((-1)^j z) \begin{cases} z_1^{-1}(1-z_2)^2, & \text{if } \; j=(1,0), \\ z_1^{-1} z_2^{-1}(1-z_1)^2, & \text{if } \; j=(0,1), \\ z_2^{-1}(1-z_1 z_2)^2, & \text{if } \; j=(1,1), \end{cases}$$

which leads to the masks of 51 nonzero coefficients for our prewavelets versus 91 nonzero coefficients for the prewavelets constructed in [10].

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